AN EXAMPLE OF A 16-VERTEX FOLKMAN EDGE (3,4)-GRAPH WITHOUT 8-CLIQUES

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Abstract

In [6] we computed the edge Folkman number $F(3,4;8) = 16$. There we used and announced without proof that in any blue-red coloring of the edges of the graph $K_1 + C_5 + C_5 + C_5$ there is either a blue 3-clique or red 4-clique. In this paper we give a detailed proof of this fact.

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1 Introduction

Only finite non-oriented graphs without multiple edges and loops are considered. We call a p-clique of the graph G a set of p vertices each two of which are adjacent. The largest positive integer p such that G contains a p-clique is denoted by $\mathrm{cl}(G)$. A set of vertices of the graph G none two of which are adjacent is called an independent set. In this paper we shall also use the following notations:

- $V(G)$ is the vertex set of the graph G;
- $E(G)$ is the edge set of the graph G;
- $N(v)$, $v \in V(G)$ is the set of all vertices of G adjacent to v;
- $G[V]$, $V \subseteq V(G)$ is the subgraph of G induced by V;
- $\chi(G)$ is the chromatic number of G;
- K_n is the complete graph on n vertices;
- C_n is the simple cycle on v vertices.

The equality $C_n = v_1 v_2 \ldots v_n$ means that $V(C_n) = \{v_1, \ldots, v_n\}$ and

$$E(C_n) = \{[v_i, v_{i+1}], i = 1, \ldots, n-1\} \cup \{[v_1, v_n]\}$$

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$ where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

Let G and H be two graphs. We shall say that H is a subgraph of G and we shall denote $H \subseteq G$ when $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.1. A 2-coloring

$$E(G) = E_1 \cup E_2, \qquad E_1 \cap E_2 = \emptyset, \tag{1.1}$$

is called a blue-red coloring of the edges of the graph G (the edges in E_1 are blue and the edges in E_2 are red).

We define for blue-red coloring (1.1) and for an arbitrary vertex $v \in V(G)$

$$\begin{aligned} &N_i(v) = \{x \in N(v) \mid [v, x] \in E_i\}, \quad i = 1, 2; \\ &G_i(v) = G[N_i(v)]. \end{aligned}$$

Definition 1.2. Let H be a subgraph of G. We say that H is a monochromatic subgraph in the blue-red coloring (1.1) if $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$. If $E(H) \subseteq E_1$ we say that H is a blue subgraph and if $E(H) \subseteq E_2$ we say that H is a red subgraph.

Definition 1.3. The blue-red coloring (1.1) is called (p, q)-free, if there are no blue p-cliques and no red q-cliques. The symbol $G \to (p, q)$ means that any blue-red coloring of $E(G)$ is not (p, q)-free. If $G \to (p, q)$ then G is called edge Folkman (p, q)-graph.

Let p, q and r be positive integers. The Folkman number $F(p, q; r)$ is defined by the equality

$$F(p, q; r) = \min\{|V(G)| : G \to (p, q) \text{ and } \mathrm{cl}(G) < r\}.$$

In [1] Folkman proved that

$$F(p, q; r) \text{ exists } \iff r > \max\{p, q\}.$$

That is why the numbers $F(p, q; r)$ are called Folkman numbers. Only few Folkman numbers are known. An exposition of the results on the Folkman numbers was given in [6]. In [6] we computed a new Folkman number, namely $F(3, 4; 8) = 16$. This result is based upon the fact that $K_1 + C_5 + C_5 + C_5 \to (3, 4)$, which was announced without proof in [6]. In this paper we give a detailed proof of this fact. So, the aim of this paper is to prove the following

Main Theorem. *Let $G = K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$, where $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$ are copies of the 5-cycle C_5. Then $G \to (3, 4)$.*

2 Auxiliary results

Lemma 2.1. *Let $E(G) = E_1 \cup E_2$ be a $(3,4)$-free red-blue coloring of the edges of the graph G. Then:*

(a) *$G_1(v)$ is a red subgraph, $v \in V(G)$;*

(b) *$(E(G_2(v)) \cap E_1) \cup (E(G_2(v)) \cap E_2)$ is a $(3,3)$-free red-blue coloring of $E(G_2(v))$, $v \in V(G)$. Thus $G_2(v) \not\to (3,3)$.*

Proof. The statement of (a) is obvious. Assume that (b) is not true. Than, since there is no blue 3-clique, $G_2(v)$ contains a red 3-clique. This red 3-clique together with the vertex v form a red 4-clique, which is a contradiction. □

Corollary 2.1. *Let $E(G) = E_1 \cup E_2$ be a $(3,4)$-free blue-red coloring of $E(G)$. Then:*

(a) *$\mathrm{cl}(G_1(v)) \leq 3$, $v \in V(G)$;*

(b) *$\mathrm{cl}(G_2(v)) \leq 5$, $v \in V(G)$;*

(c) *$G_2(v) \not\supseteq K_3 + C_5$, $v \in V(G)$.*

Proof. The statement of (a) follows from Lemma 2.1(a). The statements of (b) and (c) follow from Lemma 2.1(b), since $K_6 \to (3,3)$, [4] and $K_3 + C_5 \to (3,3)$, [2]. □

Lemma 2.2 ([5]). *Let $G = C_5 + H$, where $V(H) = \{x, y, z\}$ and $E(H) = \{[x,y],[x,z]\}$. Let $E(G) = E_1 \cup E_2$ be a $(3,3)$-free blue-red coloring of $E(G)$. Then H is monochromatic in this coloring.*

Lemma 2.3 ([3]). *Let $G = C_5 + K_2$ and $E(G) = E_1 \cup E_2$ be a $(3,3)$-free blue-red coloring of $E(G)$ such that $E(C_5) \subseteq E_i$. Then $E(K_2) \in E_i$.*

Lemma 2.4. *Let $G = K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$, where $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$ are copies of the 5-cycle C_5 and $V(K_1) = \{a\}$. Let $E(G) = E_1 \cup E_2$ be a blue-red coloring of $E(G)$ such that $\mathrm{cl}(G_1(a)) \leq 3$ and $G_2(a) \not\to (3,3)$. Then, up to numeration of the 5-cycles $C_5^{(1)}$, $C_5^{(2)}$ and $C_5^{(3)}$ we have:*

(a) *$N_1(a) \supset V(C_5^{(1)})$ and $N_1(a) \cap V(C_5^{(2)})$ is an independent set;*

(b) *$N_2(a) \supset V(C_5^{(3)})$ and $N_2(a) \cap V(C_5^{(2)})$ is not an independent set.*

Proof. Let $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$ and $C_5^{(3)} = w_1w_2w_3w_4w_5$. We shall use the following obvious fact

$$\chi(C_5) = 3. \tag{2.1}$$

It follows from (2.1) that

$$N_1(a) \cap V(C_5^{(i)}) \text{ or } N_2(a) \cap V(C_5^{(i)}) \text{ is not an independent set, } i = 1,2,3. \tag{2.2}$$

By (2.2) and Corollary 2.1(b), at least one of the sets $N_2(a) \cap V(C_5^{(i)})$, $i = 1, 2, 3$, is an independent set. Thus, at least one of the sets $N_1(a) \cap V(C_5^{(i)})$, $i = 1, 2, 3$, is not an independent set. Without loss of generality we can assume that

$$N_1(a) \cap V(C_5^{(1)}) \text{ is not an independent set.} \tag{2.3}$$

It follows from Corollary 2.1(a) and (2.3) that $N_1(a) \cap V(C_5^{(2)}) = \emptyset$ or $N_1(a) \cap V(C_5^{(3)}) = \emptyset$. Let for example $N_1(a) \cap V(C_5^{(3)}) = \emptyset$. Then

$$N_2(a) \supset V(C_5^{(3)}). \tag{2.4}$$

We have from (2.3) and Corollary 2.1(a) that $N_1(a) \cap V(C_5^{(2)})$ is an independent set. Thus, it follows from (2.1) that $N_2(a) \cap V(C_5^{(2)})$ is not an independent set. This fact together with (2.4) and Corollary 2.1(c) give us that $N_2(a) \cap V(C_5^{(1)}) = \emptyset$. Hence, $N_1(a) \supseteq V(C_5^{(1)})$. The Lemma is proved. □

Lemma 2.5. *Let $G = K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$, where $C_5^{(i)}$, $i = 1, 2, 3$, are copies of the 5-cycle C_5. Let $E(G) = E_1 \cup E_2$ be a blue-red coloring such that some of the cycles $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$ is not monochromatic. Then this coloring is not $(3, 4)$-free.*

Proof. Let $V(K_1) = \{a\}$, $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$ and $C_5^{(3)} = w_1w_2w_3w_4w_5$. Assume the opposite, i.e. $E(G) = E_1 \cup E_2$ is $(3, 4)$-free. Then by Corollary 2.1(a) we have $\mathrm{cl}(G_1(a)) \leq 3$ and by Lemma 2.1(b) we have $G_2(a) \not\to (3, 3)$. Thus, according to Lemma 2.4 we can assume that

$$N_1(a) \supseteq V(C_5^{(1)}) \text{ and } N_1(a) \cap V(C_5^{(2)}) \text{ is independent;} \tag{2.5}$$

$$N_2(a) \supseteq V(C_5^{(3)}) \text{ and } N_2(a) \cap V(C_5^{(2)}) \text{ is not independent.} \tag{2.6}$$

It follows from (2.5) and Lemma 2.1(a) that

$$E(C_5^{(1)}) \subseteq E_2. \tag{2.7}$$

We have from the statement of the Lemma 2.5 that at least one of the cycles $C_5^{(i)}$, $i = 1, 2, 3$, is not monochromatic and since $E(C_5^{(1)}) \subseteq E_2$ it remains to consider the following two cases:

Case 1. $C_5^{(2)}$ is not monochromatic. Let for example $[u_1, u_5] \in E_1$ and $[u_1, u_2] \in E_2$. If $u_1, u_2, u_5 \in N_2(a)$ by (2.6) we have $G_2(a) \supset C_2^{(3)} + G[u_1, u_2, u_5]$. It follows from Lemma 2.2 that $G_2(a)$ contains a monochromatic 3-clique. This contradicts Lemma 2.1(b). So, at least one of the vertices u_1, u_2, u_5 belongs to $N_1(a)$. Therefore, we have the following subcases:

Subcase 1a. $u_1 \in N_1(a)$. Since there are no blue 3-cliques it follows from (2.5) that

$$N_2(u_1) \supset V(C_5^{(1)}). \tag{2.8}$$

As $[u_1,a]$, $[u_1,u_5] \in E_1$ and $\mathrm{cl}(G_1(u_1)) \le 3$ (see Corollary 2.1(a)), the set $N_1(u_1) \cap V(C_5^{(3)})$ is independent. Therefore, $N_2(u_1) \cap V(C_5^{(3)})$ is not independent. This fact together with $[u_1,u_2] \in E_2$ and (2.8) give us $G_2(u_1) \supset K_3 + C_5^{(1)}$, which contradicts Corollary 2.1(c).

Subcase 1b. $u_2 \in N_1(a)$ and $u_1 \in N_2(a)$. Since there are no blue 3-cliques it follows from (2.5) that

$$N_2(u_2) \supset V(C_5^{(1)}). \tag{2.9}$$

If $N_2(u_1) \cap V(C_5^{(1)})$ contains two adjacent vertices then these vertices together with u_1 and u_2 form a red 4-clique according to (2.7) and (2.9). Hence, $N_2(u_1) \cap V(C_5^{(1)})$ is independent and, therefore, $N_1(u_1) \cap V(C_5^{(1)})$ is not independent. Since $u_5 \in N_1(u_1)$ and $\mathrm{cl}(G_1(u_1)) \le 3$ (see Corollary 2.1(a)) we have $N_1(u_1) \cap V(C_5^{(3)}) = \emptyset$. Hence

$$N_2(u_1) \supset V(C_5^{(3)}). \tag{2.10}$$

By (2.6) and (2.10)

$$V(C_5^{(3)}) \subseteq N_2(u_1) \cap N_2(a).$$

Since $[a,u_1] \in E_2$ and there are no red 4-cliques we obtain that

$$E(C_5^{(3)}) \subseteq E_1. \tag{2.11}$$

As there are no blue 3-cliques from (2.11) it follows that $N_1(u_2) \cap V(C_5^{(3)})$ is independent. Therefore, $N_2(u_1) \cap V(C_5^{(3)})$ contains two adjacent vertices. This fact together with $[u_1,u_2] \in E_2$ and (2.9) give us $G_2(u_2) \supset K_3 + C_5^{(1)}$, which contradicts Corollary 2.1(c).

Subcase 1c. $u_5 \in N_1(a)$ and $u_1, u_2 \in N_2(a)$. Since $a, u_2 \in N_2(u_1)$, it follows from Corollary 2.1(b) that at least one of the sets $N_2(u_1) \cap V(C_5^{(1)})$ and $N_2(u_1) \cap V(C_5^{(3)})$ is independent. Hence at least one of the sets $N_1(u_1) \cap V(C_5^{(1)})$, $N_1(u_1) \cap V(C_5^{(3)})$, is not independent. Assume that $N_1(u_1) \cap V(C_5^{(1)})$ is not independent. This fact together with $u_5 \in N_1(u_1)$ and Corollary 2.1(a) imply

$$N_2(u_1) \supset V(C_5^{(3)}). \tag{2.12}$$

As $[a,u_1,u_2]$ is a red 3-clique and $[a,u_1,u_2,w_i]$ is not a red 4-clique, $i = 1,\dots,5$, it follows from (2.6) and (2.12) that $[u_2,w_i] \in E_1$, $i = 1,\dots,5$, i.e. $N_1(u_2) \supset V(C_5^{(3)})$. We have from Lemma 2.1(a) that $E(C_5^{(3)}) \subseteq E_2$. Thus, according to (2.6) and (2.12), the vertices a and u_1 together with two adjacent vertices of $C_5^{(3)}$ form a red 4-clique, which is a contradiction.

Let us now consider the situation when $N_1(u_1) \cap V(C_5^{(3)})$ is not independent. Corollary 2.1(a) and $u_5 \in N_1(u_1)$ imply

$$N_2(u_1) \supset V(C_5^{(1)}). \tag{2.13}$$

If $N_2(u_1) \cap V(C_5^{(3)}) \neq \emptyset$ then from $a, u_2 \in N_2(u_1)$ and (2.13) it follows that $G_2(u_1) \supset K_3 + C_5^{(1)}$, which contradicts the Corollary 2.1(c). Hence $N_2(u_1) \cap V(C_5^{(3)}) = \emptyset$, i.e.

$$N_1(u_1) \supset C_5^{(3)}. \tag{2.14}$$

Since there are no blue 3-cliques we obtain from (2.14) and Lemma 2.1(a) that

$$E(C_5^{(3)}) \subseteq E_2. \tag{2.15}$$

If $N_2(u_2) \cap V(C_5^{(3)})$ is not independent then according to (2.6) and (2.15) an edge in $N_2(u_2) \cap V(C_5^{(3)})$ together with a and u_2 form a red 4-clique. Let $N_2(u_2) \cap V(C_5^{(3)})$ be independent. Then $N_1(u_2) \cap V(C_5^{(3)})$ is not independent. Thus, it follows from Corollary 2.1(a) that $N_1(u_2) \cap V(C_5^{(1)})$ is independent and $N_2(u_2) \cap V(C_5^{(1)})$ is not independent. Then an edge in $N_2(u_2) \cap V(C_5^{(1)})$ together with the vertices u_1 and u_2 form a red 4-clique, according to (2.7) and (2.13), which is a contradiction.

Case 2. $C_5^{(3)}$ is not monochromatic but $C_5^{(2)}$ is monochromatic. Without loss of generality we can assume that $[w_1, w_5] \in E_1$ and $[w_1, w_2] \in E_2$. Since $a, w_2 \in N_2(w_1)$ it follows from Corollary 2.1(b) that at least one of the sets $N_2(w_1) \cap V(C_5^{(1)})$ and $N_2(w_1) \cap V(C_5^{(2)})$ is independent. Hence at least one of the sets $N_1(w_1) \cap V(C_5^{(1)})$, $N_1(w_1) \cap V(C_5^{(2)})$ is not independent. We shall consider these possibilities:

Subcase 2a. $N_1(w_1) \cap V(C_5^{(1)})$ is not independent. Since $[w_1, w_5] \in E_1$ it follows from Corollary 2.1(a) that $N_1(w_1) \cap V(C_5^{(2)}) = \emptyset$, i.e.

$$N_2(w_1) \supset V(C_5^{(2)}). \tag{2.16}$$

By Lemma 2.1(b) $G_2(w_1)$ does not contain a monochromatic 3-clique and $G_2(w_1) \supset C_5^{(2)} + [a, w_2]$. Since $C_5^{(2)}$ is monochromatic and $[a, w_2] \in E_2$, it follows from Lemma 2.3 that

$$E(C_5^{(2)}) \subseteq E_2. \tag{2.17}$$

We see from (2.6), (2.16) and (2.17) that the vertices a and w_1 together with an edge of $C_5^{(2)}$ form a red 4-clique which is a contradiction.

Subcase 2b. $N_1(w_1) \cap V(C_5^{(2)})$ is not independent. Since $w_5 \in N_1(w_1)$ it follows from Corollary 2.1(a) that

$$N_2(w_1) \supset V(C_5^{(1)}). \tag{2.18}$$

Corollary 2.1(c) and $G_2(w_1) \supset C_5^{(1)} + [a, w_2] = K_2 + C_5^{(1)}$ imply

$$N_1(w_1) \supset V(C_5^{(2)}). \tag{2.19}$$

Lemma 2.1(a) and (2.19) give

$$E(C_5^{(2)}) \subseteq E_2. \tag{2.20}$$

Since there are no blue 3-cliques and $[w_1, w_5] \in E_1$ it follows from (2.19) that

$$N_2(w_5) \supset V(C_5^{(2)}). \tag{2.21}$$

We see from (2.6), (2.20) and (2.21) that the vertices a and w_5 together with an edge of $C_5^{(2)}$ form a red 4-clique which is a contradiction. $\square$

3 A property of the graph $C_5 + C_5 + C_5$

Let $G = C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$ where $C_5^{(i)}$, $i = 1, 2, 3$, are copies of the 5-cycle C_5. Let us consider the blue-red coloring where $E_1 = E(C_5^{(1)}) \cup E(C_5^{(2)}) \cup E(C_5^{(3)})$. It is clear that this coloring is $(3, 4)$-free. Thus $G \not\to (3, 4)$. However the following theorem holds:

Theorem 3.1. *Let $G = C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$ where $C_5^{(i)}$, $i = 1, 2, 3$, are copies of the* 5*-cycle C_5. Let $E(G) = E_1 \cup E_2$ be a blue-red coloring such that $E(C_5^{(1)}) \subseteq E_2$, $E(C_5^{(2)}) \subseteq E_1$ and $E(C_5^{(3)}) \subseteq E_1$. Then this coloring is not* $(3, 4)$*-free.*

Proof. Assume the opposite, i.e. that there are no blue 3-cliques and no red 4-cliques. Let $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$, $C_5^{(3)} = w_1w_2w_3w_4w_5$. Since the cycles $C_5^{(2)}$ and $C_5^{(3)}$ are blue and there are no blue 3-cliques we have that the sets $N_1(v_i) \cap V(C_5^{(2)})$ and $N_1(v_i) \cap V(C_5^{(3)})$ are independent. Thus, we have

$$|N_2(v_i) \cap V(C_5^{(2)})| \geq 3, \quad |N_2(v_i) \cap V(C_5^{(3)})| \geq 3, \quad i = 1, \dots, 5. \tag{3.1}$$

It follows from (3.1) that

$$N_2(x) \cap N_2(y) \cap V(C_5^{(i)}) \neq \emptyset, \quad i = 2, 3, \quad x, y \in V(C_5^{(1)}). \tag{3.2}$$

Let $x, y \in V(C_5^{(1)})$. We define

$$B_1(x, y) = \{v \in V(C_5^{(2)}) \mid [x, v], [y, v] \in E_2\},$$
$$B_2(x, y) = \{v \in V(C_5^{(3)}) \mid [x, v], [y, v] \in E_2\}.$$

We see from (3.2) that

$$B_i(x, y) \neq \emptyset, \quad i = 1, 2, \quad x, y \in V(C_5^{(1)}). \tag{3.3}$$

We shall prove that

$$\text{If } [x, y] \in E(C_5^{(1)}) \text{ then } B_i(x, y) \text{ is independent, } i = 1, 2. \tag{3.4}$$

Assume the opposite and let for example $u', u'' \in B_1(x, y)$ and $[u', u''] \in E(C_5^{(2)})$. By (3.3) there exists $w \in B_2(x, y)$. Since there are no blue 3-cliques

then at least one of the edges $[u', w]$, $[u'', w]$ is red. Hence $[x, y, u', w]$ or $[x, y, u'', w]$ is a red 4-clique, which is a contradiction.

Let u' and u'' be adjacent vertices in $C_5^{(2)}$. Since $[u', u''] \in E_1$ and there are no blue 3-cliques we have

$$N_1(u') \cap N_1(u'') \cap V(C_5^{(1)}) = \emptyset.$$

Thus $|N_1(u') \cap V(C_5^{(1)})| \le 2$ or $|N_1(u'') \cap V(C_5^{(1)})| \le 2$. Hence

$$|N_2(u') \cap V(C_5^{(1)})| \ge 3 \text{ and } |N_2(u'') \cap V(C_5^{(1)})| \ge 3. \tag{3.5}$$

So, (3.5) holds for every two adjacent vertices in $C_5^{(2)}$. Hence $|N_2(u) \cap V(C_5^{(1)})| \ge 3$ holds for at least three vertices in $C_5^{(2)}$. Thus, there exist two adjacent vertices in $C_5^{(2)}$, for example u_1 and u_2, such that

$$|N_2(u_1) \cap V(C_5^{(1)})| \ge 3 \text{ or } |N_2(u_2) \cap V(C_5^{(1)})| \ge 3. \tag{3.6}$$

If the both inequalities in (3.6) are strict then $N_2(u_1) \cap N_2(u_2) \cap V(C_5^{(1)})$ contains two adjacent vertices v' and v''. Since $u_1, u_2 \in B(v', v'')$ then this contradicts (3.4). Thus, we may assume that $|N_2(u_1) \cap V(C_5^{(1)})| = 3$. Hence $N_2(u_1) \cap V(C_5^{(1)})$ contains two adjacent vertices, for example v_3 and v_4. Now we shall prove that the third vertex in $N_2(u_1) \cap V(C_5^{(1)})$ is the vertex v_1. Assume the opposite. Then $v_2 \in N_2(u_1) \cap V(C_5^{(1)})$ or $v_5 \in N_2(u_1) \cap V(C_5^{(1)})$. Let $v_2 \in N_2(u_1) \cap V(C_5^{(1)})$. Then $v_1, v_5 \in N_1(u_1)$. Since $v_1, v_5, u_2 \in N_1(u_1)$ it follows from Corollary 2.1(a) that $N_1(u_1) \cap V(C_5^{(3)}) = \emptyset$. Thus, $G_2(u_1)$ contains $C_5^{(3)} + [v_3, v_4]K_2 + C_5$. According to Lemma 2.1(b) $G_2(u_1)$ does not contain monochromatic 3-cliques. As $E(C_5^{(3)}) \subseteq E_1$ and $[v_3, v_4] \in E_2$, this contradicts Lemma 2.3. We proved that $v_2 \notin N_2(u_1)$. Analogously we prove that $v_5 \notin N_2(u_1)$. So,

$$v_1, v_3, v_4 \in N_2(u_1) \text{ and } v_2, v_5 \in N_1(u_1). \tag{3.7}$$

By (3.3) we can assume that $w_1 \in B_2(v_3, v_4)$. Since $[v_3, v_4, u_1, w_1]$ is not a red 4-clique we have

$$[u_1, w_1] \in E_1. \tag{3.8}$$

As there are no blue 3-cliques and $[u_1, v_2]$, $[u_1, v_5] \in E_1$, it follows that $[w_1, v_2]$, $[w_1, v_5] \in E_2$. Taking into consideration $w_1 \in B_2(v_3, v_4)$ we have

$$[w_1, v_i] \in E_2, \quad i = 2, 3, 4, 5. \tag{3.9}$$

By (3.3) there is $u \in B_1(v_2, v_3)$. Since $[v_2, u_1] \in E_1$ then $u \neq u_1$. We shall prove that $u = u_3$ or $u = u_4$. Assume the opposite. Then $u = u_2$ or $u = u_5$. Let, for example, $u = u_2$. Since $[v_2, v_3, u_2, w_1]$ is not a red 4-clique, it follows from (3.9) and $u_2 \in B_1(v_2, v_3)$ that $[u_2, w_1] \in E_1$. We obtained the blue 3-clique $[u_1, u_2, w_1]$

which is a contradiction. This contradiction proves that $u = u_3$ or $u = u_4$. We can assume without loss of generality that $u = u_3$. We have

$$[u_3, w_1] \in E_1, \tag{3.10}$$

because $[v_2, v_3, u_3, w_1]$ is not a red 4-clique. By (3.3) there exists $u \in B_1(v_4, v_5)$. Repeating the above considerations about $u \in B_1(v_2, v_3)$ we see that $u = u_3$ or $u = u_4$.

Case 1. $u = u_4$. Since $[v_4, v_5, w_1, u_4]$ is not a red 4-clique, we have $[u_4, w_1] \in E_1$. Hence $[u_3, u_4, w_1]$ is a blue 3-clique, which is a contradiction.

Case 2. $u = u_3$. In this case we have $u_3 \in B_1(v_2, v_3) \cap B_1(v_4, v_5)$, i.e.

$$[u_3, v_i] \in E_2, \quad i = 2, 3, 4, 5. \tag{3.11}$$

As $[v_1, w_1, u_3]$ is not a blue 3-clique, it follows from (3.10) that $[v_1, u_3] \in E_2$ or $[v_1, w_1] \in E_2$.

Subcase 2a. $[v_1, u_3] \in E_2$. By (3.11) $N_2(u_3) \supset C_5^{(1)}$. Since there are no blue 3-cliques $N_2(u_3)$ contains two adjacent vertices $w', w'' \in V(C_5^{(3)})$. Thus $G_2(u_3) \supset C_5^{(1)} + [w', w'']$. By Lemma 2.1(b) $G_2(u_3)$ contains no monochromatic 3-cliques. This contradicts Lemma 2.3 because $E(C_5^{(1)}) \subseteq E_2$ and $[w', w''] \in E_1$.

Subcase 2b. $[v_1, w_1] \in E_2$. By (3.9) we see that $N_2(w_1) \supset V(C_5^{(1)})$. Since there are no blue 3-cliques $N_2(w_1)$ contains two adjacent vertices $u', u'' \in V(C_5^{(2)})$. Hence $N_2(w_1) \supset C_5^{(1)} + [u', u'']$ which contradicts Lemma 2.3.

The theorem is proved. □

4 Proof of Main Theorem

Let $C_5^{(1)} = v_1v_2v_3v_4v_5$, $C_5^{(2)} = u_1u_2u_3u_4u_5$, $C_5^{(3)} = w_1w_2w_3w_4w_5$ and $V(K_1) = \{a\}$. Assume the opposite, i.e. there exists a $(3,4)$-free blue-red coloring $E_1 \cup E_2$ of the edges of $K_1 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)}$. By Lemma 2.4 we can assume that:

$$N_1(a) \supset V(C_5^{(1)}) \text{ and } N_1(a) \cap V(C_5^{(2)}) \text{ is independent;} \tag{4.1}$$

$$N_2(a) \supset V(C_5^{(3)}) \text{ and } N_2(a) \cap V(C_5^{(2)}) \text{ is not independent.} \tag{4.2}$$

We shall prove that

$$E(C_5^{(i)}) \subseteq E_2, \quad i = 1, 2, 3. \tag{4.3}$$

By (4.1) and Lemma 2.1(a), $E(C_5^{(1)}) \subseteq E_2$. According to Lemma 2.5 each of the 5-cycles $C_5^{(2)}$ and $C_5^{(3)}$ is monochromatic. By (4.2) $G_2(a) \supset C_5^{(3)} + e$ where $e \in E(C_5^{(2)})$. By Lemma 2.1(b) $G_2(a)$ contains no monochromatic 3-cliques. Thus, it follows from Lemma 2.3 that the edge e and the 5-cycle $C_5^{(3)}$ have the same color. Therefore, the 5-cycles $C_5^{(2)}$ and $C_5^{(3)}$ are monochromatic of the same color. Thus, it follows from Theorem 3.1 that $E(C_5^{(2)}) \subsetneq E_1$ and $E(C_5^{(3)}) \subsetneq E_1$. We proved (4.3).

Now we shall prove that

$$N_2(a) = V(C_5^{(2)}) \cup V(C_5^{(3)}). \tag{4.4}$$

Assume the opposite. Then it follows from (4.2) that $N_1(a) \cap V(C_5^{(2)}) \neq \emptyset$. Let for example $u_1 \in N_1(a) \cap V(C_5^{(2)})$, i.e. $[u_1, a] \in E_1$. We see from (4.1) that

$$[a, u_2] \in E_2. \tag{4.5}$$

As there are no blue 3-cliques by (4.1) and $[u_1, a] \in E_1$ we obtain

$$N_2(u_1) \supset V(C_5^{(1)}). \tag{4.6}$$

We see from Corollary 2.1(a) that at least one of the sets $N_2(u_2) \cap V(C_5^{(3)})$, $N_2(u_2) \cap V(C_5^{(1)})$ is not independent. If $N_2(u_2) \cap V(C_5^{(1)})$ is not independent then it follows from (4.6) and (4.3) that the vertices u_1 and u_2 together with an edge of $C_5^{(1)}$ form a red 4-clique. If $N_2(u_2) \cap V(C_5^{(3)})$ is not independent then by (4.3), (4.5) and (4.2) the vertices a and u_2 together with an edge of $C_5^{(3)}$ form a red 4-clique. This contradiction proves (4.4).

It follows from (4.4) and Lemma 2.1(b) that

$$C_5^{(2)} + C_5^{(3)} \text{ contains no monochromatic 3-cliques.} \tag{4.7}$$

Now we obtain from (4.7) and (4.3)

$$N_2(x) \cap V(C_5^{(3)}) \text{ is independent}, \quad x \in V(C_5^{(2)}); \tag{4.8}$$

$$N_2(x) \cap V(C_5^{(2)}) \text{ is independent}, \quad x \in V(C_5^{(3)}). \tag{4.9}$$

Let us note that

$$N_1(x) \cap V(C_5^{(1)}) \text{ is independent}, \quad x \in V(C_5^{(2)}) \cup V(C_5^{(3)}). \tag{4.10}$$

Indeed, let for example $x \in V(C_5^{(2)})$. By (4.8) $N_1(x) \cap V(C_5^{(3)})$ is not independent. This fact and Corollary 2.1(a) prove (4.10).

We shall prove that

$$N_1(x) \cap V(C_5^{(2)}),\ x \in V(C_5^{(1)}) \text{ is not independent} \iff N_2(x) \supset V(C_5^{(3)}); \tag{4.11}$$

$$N_1(x) \cap V(C_5^{(3)}),\ x \in V(C_5^{(1)}) \text{ is not independent} \iff N_2(x) \supset V(C_5^{(2)}). \tag{4.12}$$

The statements (4.11) and (4.12) are proved analogously. That is why we shall prove (4.11) only. Let $N_1(x) \cap V(C_5^{(2)})$, $x \in V(C_5^{(1)})$ be not independent. Since $[x, a] \in E_1$, it follows from Corollary 2.1(a) that $N_1(x) \cap V(C_5^{(3)}) = \emptyset$, i.e. $N_2(x) \supset V(C_5^{(3)})$. Let now $N_2(x) \supset V(C_5^{(3)})$, $x \in V(C_5^{(1)})$. Assume that $N_1(x) \cap V(C_5^{(2)})$ is independent. Then $N_2(x) \cap V(C_5^{(2)})$ is not independent. Since $C_5^{(1)}$ is red,

$G_2(x) \supset K_3 + C_5^{(3)}$ which contradicts Corollary 2.1(c). So, (4.11) and (4.12) are proved. Using (4.11) and (4.12) we shall prove that

$$N_1(x) \cap V(C_5^{(i)}),\ i = 2, 3, \text{ is independent, } x \in V(C_5^{(1)}). \tag{4.13}$$

Assume that (4.13) is wrong and let for example $N_1(v_1) \cap V(C_5^{(2)})$ is not independent (remind that $C_5^{(1)} = v_1v_2v_3v_4v_5$). Then by (4.11) $N_2(v_1) \supset V(C_5^{(3)})$. If $N_2(v_2) \cap V(C_5^{(3)})$ is not independent then v_1 and v_2 together with two adjacent vertices from $N_2(v_2) \cap V(C_5^{(3)})$ form a red 4-clique, which is a contradiction. Therefore, $N_1(v_2) \cap V(C_5^{(3)})$ is not independent. Thus (4.12) gives $N_2(v_2) \supset V(C_5^{(2)})$. Repeating the above considerations about the vertex v_1 on v_2 we obtain $N_2(v_3) \supset V(C_5^{(3)})$. In the same way it follows from $N_2(v_3) \supset V(C_5^{(3)})$ that $N_2(v_4) \supset V(C_5^{(2)})$. At the end it follows from $N_2(v_4) \supset V(C_5^{(2)})$ that $N_2(v_5) \supset V(C_5^{(3)})$. So, we proved that

$$N_2(v_1) \cap N_2(v_5) \supset V(C_5^{(3)}).$$

Thus, it follows from (4.3) that v_1 and v_5 together with an edge of $C_5^{(3)}$ form a red 4-clique, which is a contradiction. This contradiction proves (4.13). According to (4.13) it follows from (4.11) and (4.12) that

$$N_2(x) \not\supset V(C_5^{(i)}),\ i = 2, 3, \quad x \in V(C_5^{(1)}). \tag{4.14}$$

Let $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. By (4.10) $|N_1(x) \cap V(C_5^{(1)})| \leq 2$. Thus, we have the following possibilities:

Case 1. $N_1(x) \cap V(C_5^{(1)}) = \emptyset$ for some vertex $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. Let for example $N_1(u_1) \cap V(C_5^{(1)}) = \emptyset$ (remind that $C_5^{(2)} = u_1u_2u_3u_4u_5$). Then $N_2(u_1) \supset V(C_5^{(1)})$. We have from (4.10) that $N_2(u_2) \cap V(C_5^{(1)})$ is not independent. Thus u_1 and u_2 together with two adjacent vertices from $N_2(u_2) \cap V(C_5^{(1)})$ form a red 4-clique, which is a contradiction.

Case 2. $|N_1(x) \cap V(C_5^{(1)})| = 1$ for some vertex $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. Let for example $|N_1(u_1) \cap V(C_5^{(1)})| = 1$. Without loss of generality we can consider that $[u_1, v_1] \in E_1$ and $[u_1, v_i] \in E_2$, $i = 2, 3, 4, 5$. According to (4.14) we can assume that $[v_1, w_1] \in E_1$. Since there are no blue 3-cliques, $[u_1, w_1] \in E_2$. It follows from (4.10) that $N_2(w_1) \cap V(C_5^{(1)})$ contains two adjacent vertices. As

$$N_2(w_1) \cap V(C_5^{(1)}) \subseteq N_2(u_1) \cap V(C_5^{(1)}) = \{v_2, v_3, v_4, v_5\}$$

we see that u_1 and w_1 together with two adjacent vertices in $\{v_2, v_3, v_4, v_5\}$ form a red 4-clique, which is a contradiction.

Case 3. $|N_1(x) \cap V(C_5^{(1)})| = 2$ for every $x \in V(C_5^{(2)}) \cup V(C_5^{(3)})$. According to (4.8) $N_1(u_1) \cap V(C_5^{(3)})$ is not independent. Thus, we can assume that $w_1, w_2 \in N_1(u_1) \cap V(C_5^{(3)})$, i.e.

$$[u_1, w_1], [u_1, w_2] \in E_1. \tag{4.15}$$

It follows from (4.13)

$$N_1(w_1) \cap N_1(w_2) \cap V(C_5^{(1)}) = \emptyset. \tag{4.16}$$

In the considered case we have

$$|N_1(w_1) \cap V(C_5^{(1)})| = |N_1(w_2) \cap V(C_5^{(1)})| = |N_1(u_1) \cap V(C_5^{(1)})| = 2.$$

We obtain from (4.16)

$$N_1(u_1) \cap N_1(w_1) \cap V(C_5^{(1)}) \neq \emptyset \text{ or } N_1(u_1) \cap N_1(w_2) \cap V(C_5^{(1)}) \neq \emptyset.$$

By (4.15) there is a blue 3-clique, which is a contradiction.

Main Theorem is proved.

5 Example of Folkman edge $(3,5)$-graph without 13-cliques

Using the Main Theorem we shall prove the following

Theorem 5.1. *Let $G = K_4 + C_5^{(1)} + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$ where $C_5^{(i)}$, $i = 1, \dots, 4$, are copies of the 5-cycle C_5. Then $G \to (3,5)$.*

In order to prove Theorem 5.1 we shall need the following

Lemma 5.1. *Let $E(G) = E_1 \cup E_2$ is a $(3,5)$-free blue-red coloring of $E(G)$. Then:*

(a) *$G_1(v)$, $v \in V(G)$, is a red subgraph;*

(b) *$(E(G_2(v)) \cap E_1) \cup (E(G_2(v)) \cap E_2)$ is a $(3,4)$-free blue-red coloring of $E(G_2(v))$, $v \in V(G)$. Thus, $G_2(v) \not\to (3,4)$.*

Lemma 5.1 is proved in the same way as Lemma 2.1.

Corollary 5.1. *Let $E(G) = E_1 \cup E_2$ be a $(3,5)$-free blue-red coloring of $E(G)$. Then:*

(a) *$\mathrm{cl}(G_1(v)) \leq 4$, $v \in V(G)$;*

(b) *$\mathrm{cl}(G_2(v)) \leq 8$, $v \in V(G)$;*

(c) *$G_2(v) \not\supset K_4 + C_5 + C_5$;*

(d) *$G_2(v) \not\supset K_1 + C_5 + C_5 + C_5$.*

Proof. The statement (a) follows from Lemma 5.1(a). The statement (b) follows from Lemma 5.1(b) and $K_9 \to (3,4)$, [4]. The statement (c) follows from Lemma 5.1(b) and $K_4 + C_5 + C_5 \to (3,4)$, [8]. The statement (d) follows from Lemma 5.1(b) and Main Theorem. □

Proof of Theorem 5.1. Assume the opposite, i.e. there exists a blue-red coloring $E(G) = E_1 \cup E_2$, which is (3,5)-free. Let $V(K_4) = \{a_1, a_2, a_3, a_4\}$.

Case 1. There exists $a_i \in V(K_4)$ such that $|N_1(a_i) \cap V(K_4)| = 3$. Let for example $[a_1, a_2]$, $[a_1, a_3]$, $[a_1, a_4] \in E_1$. By Corollary 5.1(a) at most one of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, is not empty, i.e. $N_2(a_1)$ contains at least three of the cycles $C_5^{(i)}$, $i = 1,2,3,4$. Let for example

$$N_2(a_1) \supset V(C_5^{(2)}) \cup V(C_5^{(3)}) \cup V(C_5^{(4)}).$$

By Corollary 5.1(a) it follows that $N_1(a_1) \cap V(C_5^{(1)})$ is independent. Thus, $N_2(a_1) \cap V(C_5^{(1)}) \neq \emptyset$. We obtained that $G_2(a_1) \supset K_1 + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(d).

Case 2. There exists $a_i \in V(K_4)$ such that $|N_1(a_i) \cap V(K_4)| = 2$. Let for example $[a_1, a_2]$, $[a_1, a_3] \in E_1$ and $[a_1, a_4] \in E_2$. Since $[a_1, a_4] \in E_2$ if the sets $N_2(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, are not independent then $G_2(a_1) \supset K_9$, which contradicts Corollary 5.1(b). Hence, at least one of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, is not independent. Let for example $N_1(a_1) \cap V(C_5^{(1)})$ is not independent. According to Corollary 5.1(a) it follows from this fact and $[a_1, a_2]$, $[a_1, a_3] \in E_1$ that $N_1(a_1) \cap V(C_5^{(i)})\emptyset$, $i = 2,3,4$, i.e. $N_2(a_1) \supset V(C_5^{(i)})$, $i = 2,3,4$. As $[a_1, a_4] \in E_2$ we have $G_2(a_1) \supset K_1 + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(d).

Case 3. There exist $a_i \in V(K_4)$ such that $|N_1(a_i) \cap V(K_4)| = 1$. Let for example $[a_1, a_2] \in E_1$ and $[a_1, a_3]$, $[a_1, a_4] \in E_2$. We see from Corollary 5.1(a) that at least three of the sets $N_2(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, are not independent. Let for example $N_2(a_1) \cap V(C_5^{(2)})$, $N_2(a_1) \cap V(C_5^{(3)})$ and $N_2(a_1) \cap V(C_5^{(4)})$ are not independent. Since $[a_1, a_3]$, $[a_1, a_4] \in E_2$ it follows from Corollary 5.1(b) that $N_1(a_1) \supset V(C_5^{(1)})$. According to Lemma 5.1(a) it follows from this fact and $[a_1, a_2] \in E_1$ that at least two of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 2,3,4$, are empty. Therefore, we can assume that $N_2(a_1) \supset V(C_5^{(3)})$ and $N_2(a_1) \supset V(C_5^{(4)})$. Since $N_2(a_1) \cap V(C_5^{(2)})$ is not independent we have $G_2(a_1) \supset K_4 + V(C_5^{(3)}) + V(C_5^{(4)})$, which contradicts Corollary 5.1(c).

Case 4. $E(K_4) \subseteq E_2$. Since $[a_1, a_i] \in E_2$, $i = 2,3,4$, it follows from Corollary 5.1(b) that at least two of the sets $N_1(a_1) \cap V(C_5^{(i)})$, $i = 1,2,3,4$, are not independent. Let for example $N_1(a_1) \cap V(C_5^{(1)})$ and $N_1(a_1) \cap V(C_5^{(2)})$ are not independent. Then by Corollary 5.1(a) $N_1(a_1) \cap V(C_5^{(3)}) = \emptyset$ and $N_1(a_1) \cap V(C_5^{(4)}) = \emptyset$, i.e.

$$N_2(a_1) \supseteq V(C_5^{(3)}) \cup V(C_5^{(4)}). \tag{5.1}$$

Since $[a_1, a_i] \in E_2$, $i = 2,3,4$, it follows from (5.1) and Corollary 5.1(c) that $N_2(a_1) \cap V(C_5^{(1)}) = \emptyset$ and $N_2(a_1) \cap V(C_5^{(2)}) = \emptyset$. That is why, we have from (5.1)

$$N_1(a_1) = V(C_5^{(1)}) \cup V(C_5^{(2)}). \tag{5.2}$$

As the vertices a_1, a_2, a_3, a_4 are equivalent in this case the above considerations prove that

$$N_1(a_i),\ i = 1,2,3,4, \text{ is a union of two of the cycles } C_5^{(1)},\ C_5^{(2)},\ C_5^{(3)},\ C_5^{(4)}. \quad (5.3)$$

Lemma 5.1(a) and (5.2) imply

$$C_5^{(1)} + C_5^{(2)} \text{ is a red subgraph.} \quad (5.4)$$

Since there are no red 5-cliques we see from (5.4) that

$$N_1(a_i) \cap V(C_5^{(1)}) \neq \emptyset \text{ or } N_1(a_i) \cap V(C_5^{(2)}) \neq \emptyset, \quad i = 2,3,4.$$

Thus, by (5.3) we have that

$$N_1(a_i) \supset V(C_5^{(1)}) \text{ or } N_1(a_i) \supset V(C_5^{(2)}), \quad i = 2,3,4. \quad (5.5)$$

Hence, we can assume that

$$N_1(a_2) \supset V(C_5^{(1)}) \text{ and } N_1(a_3) \supset V(C_5^{(1)}). \quad (5.6)$$

Let $C_5^{(1)} = v_1v_2v_3v_4v_5$. By (5.5) we have the following possibilities:

Subcase 4a. $N_1(a_4) \supset V(C_5^{(1)})$. According to (5.6) $[v_1, a_i] \in E_1$, $i = 1,2,3,4$. Hence, by Corollary 5.1(a) $N_1(v_1) \cap V(C_5^{(i)}) = \emptyset$, $i = 2,3,4$, i.e. $G_2(v_1) \supset C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$. By (5.2) $[v_1, v_2] \in E_2$. Thus, $G_2(v_1) \supset K_1 + C_5^{(2)} + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(d).

Subcase 4b. $N_1(a_4) \cap V(C_5^{(1)}) = \emptyset$, i.e. $N_2(a_4) \supset V(C_5^{(1)})$. We have from (5.2) and (5.6) that $[v_1, a_i] \in E_1$, $i = 1,2,3$, and $[v_1, a_4] \in E_2$. By Corollary 5.1(a) at least two of the sets $N_1(v_1) \cap V(C_5^{(i)})$, $i = 2,3,4$, are empty. Thus, we can assume that

$$G_2(v_1) \supset C_5^{(3)} + C_5^{(4)}. \quad (5.7)$$

It follows from Corollary 5.1(a) that $N_1(v_1) \cap V(C_5^{(2)})$ is independent. Hence, $N_2(v_1) \cap V(C_5^{(2)})$ is not independent. This fact together with $[v_1, v_2]$, $[v_1, a_4] \in E_2$ and (5.7) gives $G_2(v_1) \supset K_4 + C_5^{(3)} + C_5^{(4)}$, which contradicts Corollary 5.1(c). This contradiction finishes the proof of Theorem 5.1. □

Since $\mathrm{cl}(G) = 12$ and $V(G)| = 24$ Theorem 5.1 implies

Corollary 5.2. $F(3,5;13) \leq 24$.

Lin proved in [7] that $F(3,5;13) \geq 18$. In [9] Nenov improved this result proving that either $K_8 + C_5 + C_5 \to (3,5)$ or $F(3,5;13) \geq 19$.

References

[1] Folkman, J. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.* **18**, 1970, 19–24.

[2] Graham, R. On edgewise 2-colored graph with monohromatic triangles containing no complete hexagon. *J. Combin. Theory*, **4**, 1968, 300.

[3] Graham, R., J. Spencer. On small graphs with forced monochromatic triangles. Recent Trends in Graph Theory, Lecture Notes in Math, **186**, 1971, 137–141.

[4] Greenwood, R., A. Gleason. Combinatorial relation and chromatic graphs. *Canad. J. Math.*, **7**, 1955, 1–7.

[5] Khadzhiivanov, N., N. Nenov. On the Graham–Spencer number. *Compt. rend. Acad. bulg. Sci.*, **31**, 1978, 631–633.

[6] Kolev, N., N. Nenov. The Folkman number $F_e(3, 4; 8)$ is equal to 16. *Compt. rend. Acad. bulg. Sci.*, **59**, 2006, No 1, 25–30.

[7] Lin S. On Ramsey number and K_r-coloring of graphs. *J. Comb. Theory*, **B12**, 1972, 82–92.

[8] Nenov, N. On the Ramsey $(3, 4)$-graphs. *Ann. Univ. Sofia Fac. Math. Mech.*, **73**, 1979, 185–190 (1986) (in Russian).

[9] Nenov, N. On the Zykov numbers and some its applications to Ramsey theory. *Serdica Bulg. Math. Publ.*, **9**, 1983, 161–167 (in Russian).

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